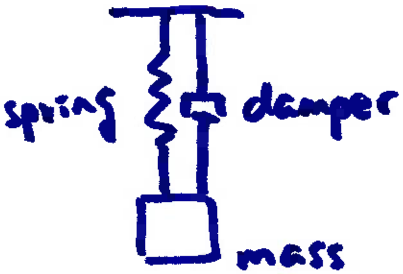
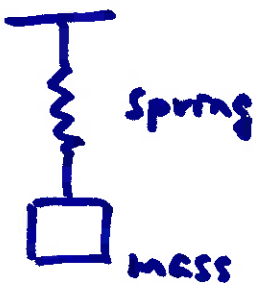
3.7 Mechanical Vibrations

mass-spring-damper problem



for now, let's leave the damper out



motion of mass traces out a sine or cosine

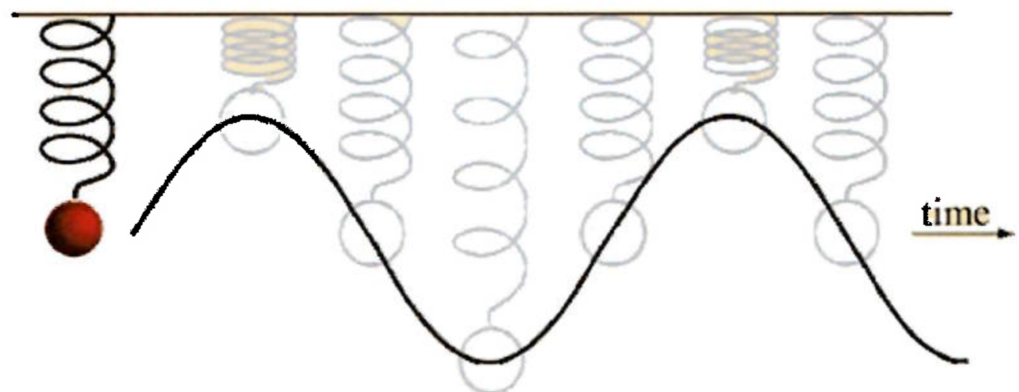
why?

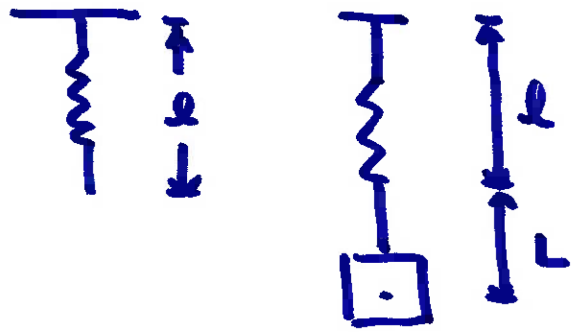
Hooke's Law (spring force) : $|F_s| = KL$

spring constant

elongation with respect to the natural length of the spring

natural length: length of spring undisturbed → no force


time



l: natural length

L: elongation

$L > 0 \rightarrow$ stretched

$L < 0 \rightarrow$ compressed

direction of force of the spring is dependent on L

forces on the block :

spring $|F_s| = KL$

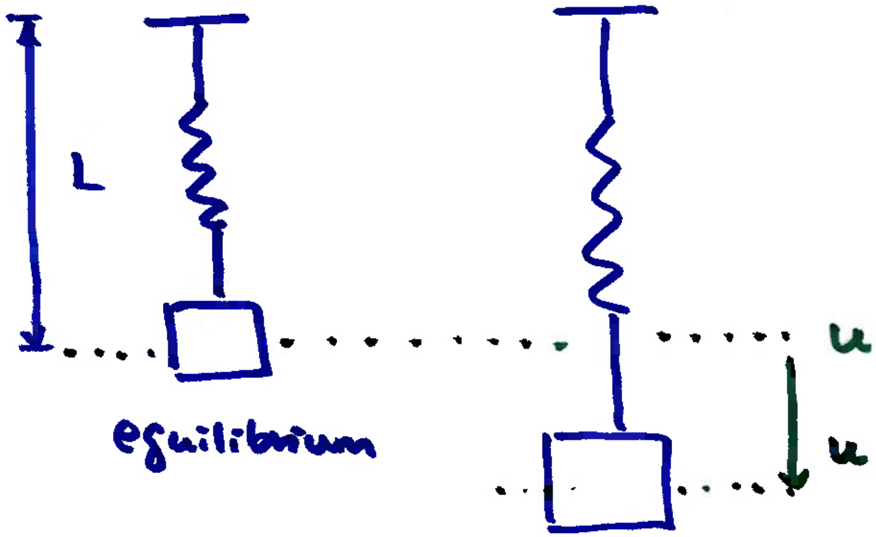
weight $= mg$

at equilibrium (leave the block hanging w/o stretching or compressing further)

→ forces balance out

→ $\boxed{KL = mg}$

if we stretch it further



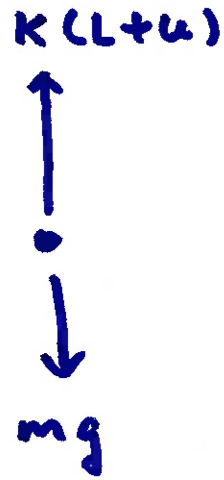
$u=0$ (length at equilibrium) with respect to

equilibrium

stretched more

forces:

length at equilibrium



define u as <u>positive</u> <u>down</u>

$|F_s| = k(L+u)$ if $u > 0 \to F_s < 0$ (spring pulls up)

$u < L$ (compress) $\to F_s > 0$ (spring pushes down)

to account for this, we can just use $F_s = -k(L+u)$

now we use <u>Newton's 2nd Law</u> to describe the motion

$$\text{force} = \underbrace{\text{mass} \cdot \text{acceleration}}$$

$$mu'' = \underset{\text{weight}}{mg} - \underset{\text{spring}}{k(L+u)}$$

$$mu'' = mg - kL - ku$$

from ~~each~~ earlier, we found $mg = kL$

So the equation of motion is

$$\boxed{mu'' + ku = 0}$$

initial conditions

$u(0) = u_0$ initial elongation

$u'(0) = v_0$ initial velocity

down is positive

Example mass weighs 10 lb stretches a spring by 6 in.
If the mass is pulled down a distance of 2 in
from the equilibrium and set in motion with
an upward velocity of 1 ft/s.
Describe the motion $u(t)$.

1st sentence: equilibrium condition

$\rightarrow L = 6 \text{ in} = \frac{1}{2} \text{ ft}$

Hooke's Law: $|F_s| = kL$

$10 \text{ lb} = k \cdot \frac{1}{2}$
(weight)

$$\boxed{k = 20 \text{ lb/ft}}$$

$mu'' + ku = 0$

mass!

mass $\neq$ weight

weight $= m \cdot g = 10$ lb

g in English units is $32\ \text{ft}/\text{s}^2$

(in SI units is $9.8\ \text{m}/\text{s}^2$)

$$m = \frac{10}{32} = \frac{5}{16}$$

$$\frac{5}{16} u'' + 20u = 0$$

$$\boxed{u'' + 64u = 0}$$

$$\boxed{u(0) = \tfrac{1}{6}}$$

"mass is pulled down 2 in" $2\text{ in} = \frac{1}{6}$ ft

$$\boxed{u'(0) = -1}$$

negative because it's upward (down is positive)

$r^2 + 64 = 0 \quad r = \pm 8i$

$$u(t) = C_1 \cos(8t) + C_2 \sin(8t)$$

with initial conditions we can find C_1, C_2

$\vdots$

$$\boxed{u(t) = \tfrac{1}{6}\cos(8t) - \tfrac{1}{8}\sin(8t)}$$

but it's much easier to analyze it in this form:

$$u(t) = R\cos(\omega_0 t - \delta)$$

phase shift

angular frequency (how fast)

amplitude

(how much up/down)

how to convert $u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

to $R\cos(\omega_0 t - \delta)$?

identity: $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

$$R\cos(\omega_0 t - \delta)$$

$$= R\cos(\omega_0 t)\overbrace{\cos(-\delta)}^{\cos(\delta)} - R\sin(\omega_0 t)\overbrace{\sin(-\delta)}^{-\sin(\delta)}$$

$$= R\cos(\delta)\cos(\omega_0 t) + R\sin(\delta)\sin(\omega_0 t)$$